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(In)validity of large N orientifold equivalence

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ABSTRACT: It has been argued that the bosonic sectors of supersymmetric $SU(N)$ Yang-Mills theory, and of QCD with a single fermion in the antisymmetric (or symmetric) tensor representation, are equivalent in the $N \rightarrow \infty$ limit. If true, this correspondence can provide useful insight into properties of real QCD (with fundamental representation fermions), such as predictions [with $O(1/N)$ corrections] for the non-perturbative vacuum energy, the chiral condensate, and a variety of other observables. Several papers asserting to have proven this large N “orientifold equivalence” have appeared. By considering theories compactified on $\mathbb{R}^3 \times S^1$, we show explicitly that this large N equivalence fails for sufficiently small radius, where our analysis is reliable, due to spontaneous symmetry breaking of charge conjugation symmetry in QCD with an antisymmetric (or symmetric) tensor representation fermion. This theory is also chirally symmetric for small radius, unlike super-Yang-Mills. The situation is completely analogous to large- N equivalences based on orbifold projections: simple symmetry realization conditions are both necessary and sufficient for the validity of the large N equivalence. Whether these symmetry realization conditions are satisfied depends on the specific non-perturbative dynamics of the theory under consideration. Unbroken charge conjugation symmetry is necessary for validity of the large N orientifold equivalence. Whether or not this condition is satisfied on \mathbb{R}^4 (or $\mathbb{R}^3 \times S^1$ for sufficiently large radius) is not currently known.

KEYWORDS: $1/N$ Expansion, Spontaneous Symmetry Breaking.

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1. Introduction and results

Non-perturbative equivalences between different gauge theories, in the limit of a large rank N of the gauge group, can provide valuable insight into the dynamics of strongly coupled theories. Examples of such equivalences include volume independence [1–5], relations between $U(N)$, $SO(N)$ and $Sp(N)$ theories [6], relations between lattice theories with mixed fundamental/adjoint actions [7–9], and equivalences between theories related by orbifold projections [10–16]. In particular, equivalences relating supersymmetric and non-supersymmetric theories which are confining and asymptotically free hold promise for allowing one to convert knowledge about supersymmetric theories into improved understanding of QCD or QCD-like theories.

There has been much discussion in the string theory literature of orbifold and orientifold projections relating maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills theory to various less supersymmetric or non-supersymmetric theories. (See, for example, Refs. [17–20].) This prompted the observation that, for a wide class of projections, planar diagrams of “parent” and “daughter” gauge theories are identical in the $N \rightarrow \infty$ limit [10], implying coinciding perturbative expansions of the two theories. This planar diagram equivalence is essentially kinematic, and does not depend on supersymmetry, conformal symmetry, or any detailed dynamical properties of the theories. Several authors [11–13] suggested that a genuine non-perturbative large N equivalence may exist between theories related by orbifold projections. However, necessary and sufficient conditions for the validity of such a non-perturbative equivalence were not clear. Together with P. Kovtun, we recently demonstrated that, for a wide class of projections, validity of large N equivalence between parent and daughter theories depends *only* on certain

symmetry realizations conditions [15]. Comparison of the $N = \infty$ dynamics generated by suitable gauge invariant coherent states shows that the dynamics within the “neutral” sectors of the parent and daughter theories coincides in the $N \rightarrow \infty$ limit. Here, neutral operators in the parent theory are gauge invariant single-trace operators which are invariant under the discrete symmetries used to define the projection, while neutral operators in the daughter theory are single-trace gauge invariant operators which are also invariant under global symmetries in the daughter which are remnants of gauge symmetries in the parent theory.¹ This equivalence of $N = \infty$ dynamics within respective neutral sectors implies simple relations between connected correlation functions (as well as expectation values) of corresponding neutral operators, *provided* the ground (or thermal equilibrium) states of both theories lie within their neutral sectors. In other words, in order for this large N equivalence to be useful, it is both necessary and sufficient that *neither parent nor daughter theory spontaneously break the discrete symmetries which define their respective neutral sectors*.

In a series of papers, Armoni, Shifman, and Veneziano assert that there is a large N equivalence between the bosonic sectors of $\mathcal{N} = 1$ supersymmetric Yang-Mills (SYM) theory and QCD with a single Dirac fermion in the antisymmetric (or symmetric) tensor representation [21–24]. (See also Refs. [25–27].) This latter theory will be abbreviated as QCD(AS) or QCD(S) in the antisymmetric or symmetric case, respectively. The relation between these theories was termed an “orientifold equivalence” although, strictly speaking, neither theory is an orientifold projection of the other. This is an intriguing conjecture, with a variety of interesting implications which have been explored in these works. For example, validity of this equivalence implies that QCD(AS/S), to leading order in large N , has vanishing vacuum energy, a fermion condensate identical to that in $\mathcal{N} = 1$ SYM, and degenerate parity doublets. The equivalence also implies identical patterns of spontaneous symmetry breaking between the orientifold partners.

Armoni *et al.* [21, 23] argue that the orientifold equivalence, unlike large N equivalences based on orbifold projections, is free of twisted (or non-neutral) sectors, and therefore cannot fail due to unwanted symmetry breaking. They claimed to have provided a rigorous non-perturbative proof of this equivalence [28]. These authors also assert that their construction may be realized in a non-tachyonic string theory background and argue, based on conventional string theory wisdom, that this implies that the equivalence must be valid.

In this paper, we reexamine the relation between SYM and QCD(AS/S) and reach strikingly different conclusions. (Henceforth, “SYM” will always mean $\mathcal{N} = 1$ supersymmetric Yang-Mills theory.) We first discuss how both $U(N)$ SYM and $U(N)$ QCD(AS) theories may be obtained from $SO(2N)$ SYM by performing genuine projections.² Viewing the relation between $U(N)$ SYM and QCD(AS) as a possible “daughter-daughter” equivalence [in contrast to the more familiar parent-daughter orbifold equivalences] clarifies the relation between these theories. Most importantly, there *are* twisted sectors in both SYM and QCD(AS) corresponding to charge-conjugation odd states (or operators). Of course, the mere presence of a twisted

¹Non-neutral operators (or states) are often called “twisted”.

²A similar construction starting with $Sp(2N)$ SYM yields $U(N)$ SYM and QCD(S). For simplicity of presentation, we will focus on the QCD(AS) case.

sector does not imply any failure of large N equivalence, what is important is whether the symmetry defining this sector is spontaneously broken. In other words, the validity of large N “orientifold equivalence” between QCD(AS) and SYM requires unbroken charge conjugation symmetry in both theories. This condition has not previously been noted.³

Spontaneous breaking of charge conjugation symmetry in large N QCD(AS) is a non-trivial question. There is no theorem (analogous to the Vafa-Witten theorem [29] for parity symmetry) demonstrating that charge conjugation symmetry cannot be broken spontaneously. When the theory is formulated on \mathbb{R}^4 , the symmetry realization depends on the long distance, strongly coupled dynamics of the theory.⁴ One may, however, compactify on $\mathbb{R}^3 \times S^1$ with the circumference of the S^1 sufficiently small so that the theory is weakly coupled on this scale. In this regime, a reliable perturbative analysis is feasible. Choosing periodic boundary conditions for the fermions, we compute the one-loop effective potential for the Wilson line which wraps the S^1 . The imaginary part of the Wilson line is an order parameter for charge conjugation symmetry. By analyzing the minima of the effective potential, we find that QCD(AS) spontaneously breaks charge conjugation invariance, but does not break the discrete chiral symmetry of the theory.⁵

In contrast, supersymmetric Yang-Mills theory is known to spontaneously break its non-anomalous \mathbb{Z}_{2N} discrete chiral symmetry down to \mathbb{Z}_2 (corresponding to fermion number modulo two), leading to N degenerate vacuum states [32]. Every vacuum state is invariant under a (suitably defined) charge conjugation symmetry. Compactifying on $\mathbb{R}^3 \times S^1$, for any size S^1 , does not change this symmetry realization.

Therefore, at least on $\mathbb{R}^3 \times S^1$ with small radius, QCD(AS) and SYM have fundamentally different symmetry realizations. We also find directly that the leading $O(N^2)$ vacuum energy is non-vanishing for QCD(AS), unlike SYM. Therefore, when compactified with sufficiently small radius, the previously asserted large N equivalence between SYM and QCD(AS) is *false*. Completely analogous results hold for compactification on a three-torus, $T^3 \times \mathbb{R}$. This

³In the discussion [21, 28] of Armoni *et al.*, it was tacitly presumed that expectation values defined by the functional integral satisfy both charge conjugation invariance and large N factorization. However, large N factorization is only valid in equilibrium states which represent pure phases and satisfy cluster decomposition. If symmetries are spontaneously broken, then the functional integral corresponds to a mixed state which does not satisfy large N factorization. Patella [25] demonstrates coinciding strong coupling expansions in lattice regulated versions of these theories but, as discussed in Refs. [14, 15], this only implies equivalence within the phase of the lattice theory which is continuously connected to arbitrarily strong coupling and large fermion mass. It does not prove equivalence outside of this phase. In particular, due to the generic presence of large N phase transitions separating weak and strong coupling, equivalence within the strong-coupling, large-mass phase, at $N = \infty$, does not yield any information about continuum limits of these theories.

⁴Some results from lattice simulations of one-flavor QCD which, for $N = 3$, is the same as QCD(AS), have recently been reported [30]. This study used relatively small lattices and antiperiodic boundary conditions for the fermions, and did not attempt to test for spontaneous breaking of charge conjugation symmetry. Our results, presented below, show that the combination of small volume and antiperiodic boundary conditions suppresses charge conjugation symmetry breaking. It would be desirable to perform further lattice simulations of this theory and explicitly test the charge conjugation symmetry realization.

⁵Our treatment closely parallels the analysis of D. Tong [31], who examined a \mathbb{Z}_2 orbifold projection of $U(2N)$ SYM, yielding a non-supersymmetric $U(N) \times U(N)$ theory, and found that on $\mathbb{R}^3 \times S^1$ with sufficiently small radius, the non-supersymmetric daughter theory breaks the \mathbb{Z}_2 symmetry exchanging gauge group factors.

disproves any possibility of a general proof of large N orientifold equivalence (independent of spacetime dimension, spatial volume, etc.).

Imposing antiperiodic boundary conditions for fermions is the same as considering these theories at non-zero temperature. For sufficiently high temperature, we find that QCD(AS/S) has unbroken charge conjugation and chiral symmetry, but spontaneously breaks its \mathbb{Z}_2 center symmetry, indicating the deconfined nature of the high temperature phase. Supersymmetric Yang-Mills theory, at high temperatures, also has unbroken charge conjugation and chiral symmetry, while spontaneously breaking its $U(1)$ center symmetry. Therefore, large N equivalence of QCD(AS/S) and $\mathcal{N}=1$ SYM, within their respective neutral sectors, does hold at sufficiently high temperatures.

Hence, the phase diagram of QCD(AS), as a function of temperature and spatial periods, must have one (or more) phase transitions separating a high temperature charge conjugation invariant deconfined phase, in which large N equivalence to SYM is valid, from a confining small volume low temperature phase with broken charge conjugation symmetry and no large N equivalence to SYM. Whether QCD(AS) possesses a distinct low temperature large volume confining phase in which charge conjugation is unbroken, and large N equivalence to SYM is valid, is not currently known. This is precisely the same as the status of the large N equivalence based on the \mathbb{Z}_2 orbifold projection of SYM which yields a $U(N) \times U(N)$ gauge theory with a bifundamental fermion.⁶

To summarize, the status of the large N “orientifold equivalence” discussed in Refs. [21–27] is no better (or worse) than that of similar large N orbifold equivalences. For all large N equivalences based on orbifold or orientifold projections, appropriate symmetry realization conditions are an unavoidable, and non-trivial, necessary condition for a useful equivalence.

2. Orientifold projections and daughter-daughter equivalence

Let \mathcal{C} denote charge conjugation. As noted in the Introduction, QCD(AS) and $\mathcal{N}=1$ SYM are not directly related to each other by an orientifold projection.⁷ However, both these theories may be obtained by applying (different) \mathbb{Z}_2 projections to a common parent theory, namely $SO(2N)$ $\mathcal{N}=1$ super-Yang-Mills. The field content of the theory consists of a gauge boson A_μ

⁶This large N orbifold equivalence is applicable at sufficiently high temperatures, but is known to fail on $\mathbb{R}^3 \times S^1$ (with periodic boundary conditions) for sufficiently small radius, due to \mathbb{Z}_2 symmetry breaking in the daughter theory. No evidence demonstrating \mathbb{Z}_2 symmetry breaking in this specific daughter theory, in large volume, is known [16], but neither is there any proof of the absence of such symmetry breaking.

⁷We define orientifold projections as in Polchinski (vol 1), pgs. 190–192 [33]. Orientifold projections are \mathbb{Z}_2 projections based on a discrete symmetry involving charge conjugation. Starting with a complex group $U(N)$, a projection by \mathcal{C} yields the real group $SO(N)$, while a projection by \mathcal{C} combined with an antisymmetric \mathbb{Z}_2 gauge transformations yields the pseudo-real group $Sp(N)$. The neutral sector in the parent $U(N)$ theory consists of gauge-invariant \mathcal{C} -even operators, while \mathcal{C} -odd operators form the twisted sector. There also exist “reverse projections,” involving projections by suitable \mathbb{Z}_2 gauge transformations, which take a real $SO(2N)$ [or pseudoreal $Sp(2N)$] gauge group to a complex $U(N)$ group [5]. In this case, the charge conjugation symmetry of the daughter theory is a remnant of a \mathbb{Z}_2 transformation which is part of the gauge group in the parent theory. In this case, it is the (complex) daughter theory which has a twisted sector consisting of \mathcal{C} -odd operators, while the neutral sector contains only \mathcal{C} -even operators.

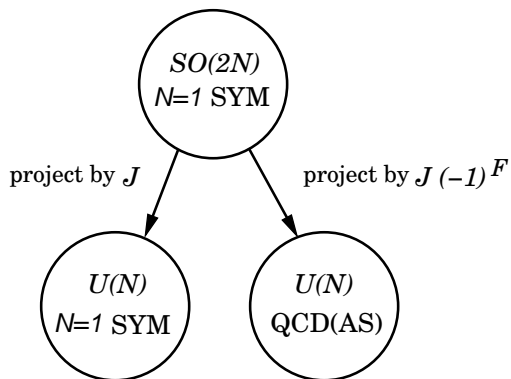


Figure 1: \mathbb{Z}_2 projections of $SO(2N)$ $\mathcal{N}=1$ super-Yang-Mills theory, yielding either $U(N)$ SYM, or $U(N)$ QCD(AS). Both daughter theories have a charge conjugation symmetry which is a remnant of a \mathbb{Z}_2 gauge transformation in the parent theory; hence both daughter theories have twisted sectors corresponding to charge conjugation odd states.

and a Majorana gluino λ . The parent theory has a discrete \mathbb{Z}_{4N-4} chiral symmetry,⁸ which is the non-anomalous remnant of $U(1)_R$ symmetry. This \mathbb{Z}_{4N-4} symmetry is spontaneously broken down to the \mathbb{Z}_2 of $(-1)^F$ (corresponding to fermion number modulo two) by the formation of a gluino condensate. Therefore, the only unbroken internal global symmetry of the parent theory is $(-1)^F$, which defines the grading into fermionic and bosonic states.

Two nontrivial \mathbb{Z}_2 projections may be applied to the parent $SO(2N)$ theory which lead to $U(N)$ gauge theories. Let $J \equiv i\sigma_2 \otimes 1_N$ denote the symplectic form which is real, anti-symmetric and an element of $SO(2N)$. If one projects by J (*i.e.*, imposes the constraints $A_\mu = J A_\mu J^T$ and $\lambda = J \lambda J^T$ on the gauge field and gluino), then the result is a $U(N)$ gauge theory with an adjoint representation Majorana fermion, which is precisely $U(N)$ $\mathcal{N}=1$ super-Yang-Mills theory. Alternatively, if one projects by J times $(-1)^F$ (corresponding to the constraints $A_\mu = J A_\mu J^T$ and $\lambda = -J \lambda J^T$), then the result is a $U(N)$ gauge theory with a Dirac fermion in the antisymmetric tensor representation, or in other words, $U(N)$ QCD(AS).⁹

We want to identify, correctly, the twisted and neutral sectors associated with each of these projections. Consider first the projection by J yielding $U(N)$ SYM (the left branch of the figure). This projection only involves an element of the gauge group, and gauge symmetries never break spontaneously. Therefore, the neutral sector in the parent $SO(2N)$ theory consists of all (gauge invariant, single-trace) operators, both fermionic and bosonic. All such operators have images in the daughter $U(N)$ theory. The daughter theory has a complex gauge group, and a charge conjugation symmetry \mathcal{C} . This charge conjugation symmetry is the image of the gauge transformation $K \equiv \sigma_3 \otimes 1_N$ in the parent $SO(2N)$ theory. This

⁸See, for example, Ref. [34], pg. 72.

⁹Similarly, if one starts with $Sp(2N)$ $\mathcal{N}=1$ SYM and performs similar \mathbb{Z}_2 projections, one may obtain either $U(N)$ $\mathcal{N}=1$ SYM or $U(N)$ QCD(S).

\mathbb{Z}_2 symmetry of the daughter is the analogue of the discrete symmetry cyclically permuting equivalent gauge group factors in quiver theories arising from orbifold constructions. The neutral sector in the daughter theory consists of all (single-trace, gauge invariant) \mathcal{C} -even operators, both bosonic and fermionic, while the twisted sector is the complementary set of all \mathcal{C} -odd operators.

Now consider the projection by $J(-1)^F$ yielding $U(N)$ QCD(AS) (the right branch of the figure). The inclusion of $(-1)^F$ in the projection modifies the twisted and neutral sectors associated with this projection. In the parent theory, gauge invariant, single-trace operators which are invariant under $J(-1)^F$ must be bosonic (*e.g.*, $\text{tr } \lambda\lambda$), while the twisted sector is composed of single-trace fermionic operators (*e.g.*, $\text{tr } \lambda\lambda\lambda_\alpha$). As in the previous case, the daughter theory has a charge conjugation symmetry \mathcal{C} which can be used to grade operators and is the image of the gauge transformation by K in the parent theory. In the daughter theory, because the fermions are in the two index antisymmetric representation there are no gauge invariant fermionic operators (*i.e.*, physical operators built from an odd number of fermion fields). Consequently, the neutral sector in the daughter now consists of \mathcal{C} -even bosonic operators, while the twisted sector is composed of \mathcal{C} -odd bosonic operators. Note that the neutral sectors, in both parent and daughter, for the projection by $J(-1)^F$ are just the bosonic subsets of the neutral sectors for the first projection by J .

In Ref. [15] we discussed how, by exploiting the infinite dimensional group structure which underlies gauge invariant coherent states, one may compare the $N = \infty$ dynamics of parent and daughter theories. In that paper, we focused on a particular class of orbifold projections which generate quiver gauge theories. However, the comparison may trivially be adapted to the projections by J , or $J(-1)^F$, of $SO(2N)$ SYM discussed above. Since there are no significant changes, we refer readers to Ref. [15] for details. One finds exactly the same conclusion for these projections: the $N = \infty$ dynamics of parent and daughter theories coincides *within their respective neutral sectors*. In other words, the large N dynamics (generated by gauge invariant, single trace operators) in $SO(2N)$ SYM coincides with the dynamics within the \mathcal{C} -even sector of $U(N)$ SYM. And the large N dynamics in the bosonic sector of $SO(2N)$ SYM coincides with the large N dynamics within the \mathcal{C} -even bosonic sector of $U(N)$ QCD(AS).

This large N equivalence between a common parent theory and two different daughter theories automatically implies an equivalence between the two daughter theories: the large N dynamics of $U(N)$ SYM and QCD(AS), within their \mathcal{C} -even bosonic sectors, coincides. This is the “orientifold equivalence” of Armoni *et al.*, and it is naturally viewed as an example of large N “daughter-daughter” equivalence. As the approach of Ref. [15] makes clear, this large N equivalence of SYM and QCD(AS), within the \mathcal{C} -even bosonic sector, is essentially kinematic. But the utility of this equivalence depends crucially on whether ground (or equilibrium) states of both theories lie within this sector. If they do, then the large N equivalence not only implies relations between expectation values of corresponding (\mathcal{C} -even, bosonic) operators, it also implies that the leading large N behavior of connected correlators of such operators coincide.¹⁰ But if charge conjugation symmetry is broken in either theory, so that the ground

¹⁰Explicitly, $\lim_{N \rightarrow \infty} (N^2)^{M-1} \langle \mathcal{O}_1 \cdots \mathcal{O}_M \rangle_{\text{conn}}^{\text{SYM}} = \lim_{N \rightarrow \infty} (N^2)^{M-1} \langle \tilde{\mathcal{O}}_1 \cdots \tilde{\mathcal{O}}_M \rangle_{\text{conn}}^{\text{QCD(AS)}}$, where $\{\mathcal{O}_i\}$ are \mathcal{C} -

(or equilibrium) state is not \mathcal{C} -invariant, then the large N equivalence within the neutral sectors generates no useful information about vacuum (or equilibrium) expectation values or correlation functions.

3. Symmetry realizations on $\mathbb{R}^3 \times S^1$ at small radius

In strongly coupled gauge theories there is, in general, no easy way to tell whether a global symmetry is spontaneously broken or not. For both QCD(AS) and $\mathcal{N}=1$ SYM, it is possible to perform lattice simulations, but so far only limited results on relatively small lattices are available [30, 35]. Larger simulations with a variety of lattice sizes will be needed to firmly establish the symmetry realization of QCD(AS) on \mathbb{R}^4 .

However, by considering this theory on $\mathbb{R}^3 \times S^1$ with the circumference L (or radius $L/2\pi$) of the S^1 circle much smaller than the inverse of the non-perturbative confinement scale Λ , we can take advantage of the asymptotic freedom of the theory. In this regime, $L \ll \Lambda^{-1}$, the gauge coupling on the scale L^{-1} is small, and perturbative methods are reliable.

One-loop effective potential

We consider $\mathcal{N}=1$ SYM and QCD(AS) on $\mathbb{R}^3 \times S^1$ with periodic boundary conditions for the fermions on the S^1 . For SYM, this choice of boundary conditions preserves supersymmetry. Let $\Omega(\mathbf{x})$ denote the group valued Wilson line wrapping the S^1 (*i.e.*, the path-ordered exponential of the line integral of the gauge field around the periodic direction), sitting at point \mathbf{x} in \mathbb{R}^3 . Minima of the classical gauge field action correspond to vanishing gauge field strength and constant but arbitrary values of Ω . At one-loop order, quantum fluctuations generate a non-trivial effective potential for Ω which we will compute.¹¹

We may work in a gauge where the gauge field is constant and the Wilson line is diagonal,

$$\Omega(\mathbf{x}) = \Omega \equiv \text{diag}(e^{iv_1}, \dots, e^{iv_N}). \quad (3.1)$$

The angles $\{v_i\}$ are periodic variables defined modulo 2π . The resulting one-loop effective potential, for either theory, may be expressed as

$$V_{\text{eff}}(\Omega) \equiv -\frac{1}{L\mathcal{V}} \ln Z[\Omega] = -\frac{1}{L\mathcal{V}} \ln \left[\frac{\det^\alpha(-D_{\mathcal{R}}^2)}{\det(-D_{\text{adj}}^2)} \right], \quad (3.2)$$

where $D_{\mathcal{R}}^2$ denotes the covariant Laplacian, in the background of a constant gauge field, for representation \mathcal{R} , and \mathcal{V} is the volume of \mathbb{R}^3 . The functional determinant in the numerator of the logarithm comes from integrating out fermions in representation \mathcal{R} , while the adjoint representation determinant in the denominator is the combined result of fluctuations in the

even single-trace bosonic operators in $U(N)$ SYM and $\{\tilde{\mathcal{O}}_i\}$ are the corresponding (\mathcal{C} -even, bosonic, single-trace) operators produced by the mappings connecting both theories to $SO(2N)$ SYM.

¹¹This exercise is a simple adaptation of the corresponding calculation for hot QCD in appendix D of Ref. [36].

gauge boson and ghost fields. The exponent α equals 1 for a Majorana fermion (relevant for $\mathcal{N}=1$ SYM), while $\alpha = 2$ for a Dirac fermion [relevant for QCD(AS)].

From the form (3.2), it is immediate that the one-loop effective potential vanishes identically for $\mathcal{N}=1$ SYM (whose Majorana fermion is in the adjoint representation),

$$V_{\text{eff}}^{\text{SYM}}(\Omega) = 0. \quad (3.3)$$

As is well known, due to supersymmetry this remains true to all orders in perturbation theory. (Non-perturbative effects generate a non-vanishing potential. This is reviewed below.)

For QCD(AS), one must compute the functional determinants in the adjoint and anti-symmetric tensor representations. Details are given in the Appendix. One finds,¹²

$$V_{\text{eff}}^{\text{QCD(AS)}}(\Omega) = \frac{1}{24\pi^2 L^4} \left\{ \sum_{i,j=1}^N [v_i - v_j]^2 (2\pi - [v_i - v_j])^2 - \frac{8}{15} \pi^4 N \right. \\ \left. - 2 \sum_{i < j=1}^N [v_i + v_j]^2 (2\pi - [v_i + v_j])^2 \right\}, \quad (3.4)$$

up to higher order corrections suppressed by g^2 . Here $[x] \equiv x \bmod 2\pi$ indicates quantities defined to lie within the interval $[0, 2\pi)$. The first sum is the contribution of gauge bosons (and ghosts), and is the same as the result one finds in hot gauge theories. The second sum is the contribution of fermions and, due to the use of periodic instead of antiperiodic boundary conditions, differs from the thermal compactification result. The $O(N)$ constant term reflects the imperfect cancellation of the zero point energies between the N^2 components of the gauge field, and the $N(N-1)$ components of the fermions. This is a subleading $O(1/N)$ correction relative to the leading $O(N^2)$ contributions of the sums.

From the result (3.4), one sees that the gluon contribution is positive definite and is minimized when the eigenvalues of Ω coincide, so that $v_i = v_j \pmod{2\pi}$. In other words, this term generates an effective attraction between eigenvalues. On the other hand, the fermionic contribution is negative definite. Global minima of the effective potential are necessarily located on the subspace where all eigenvalues coincide, $v_i = v$, since within this subspace one can simultaneously minimize every term in the first sum, and maximize every term in the second sum. Within this subspace, the potential (3.4) equals

$$V_{\text{eff}}^{\text{QCD(AS)}}(e^{iv}) = -\frac{N(N-1)}{24\pi^2 L^4} [2v]^2 (2\pi - [2v])^2 - \frac{\pi^2 N}{45 L^4}. \quad (3.5)$$

This function, which is plotted in Fig. 2, has two global minima at

$$v = \pi/2 \quad \text{and} \quad v = 3\pi/2, \quad (3.6)$$

or in other words, when $\Omega = \pm i$ (times the unit matrix). Charge conjugation acts on the Wilson line Ω as complex conjugation, or equivalently sends $v_i \rightarrow 2\pi - v_i$. The potential (3.5) is charge conjugation symmetric, but its two minima are exchanged by the action of \mathcal{C} .

¹²For QCD(S), the only difference is an additional contribution of $-2 \sum_{i=1}^N [2v_i]^2 (2\pi - [2v_i])^2 + \frac{16}{15} \pi^4 N$, inside the braces of (3.4), from the diagonal components of the symmetric representation fermions. This only affects the subleading $O(N)$ term to the effective potential and has no effect on the leading large N dynamics. Hence, the following discussion of QCD(AS) applies equally well to QCD(S).

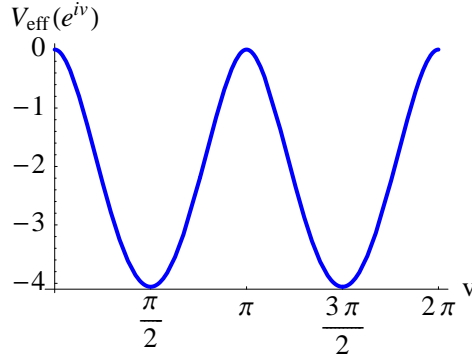


Figure 2: The one loop effective potential in QCD(AS/S), minus the irrelevant $O(N)$ additive constant and divided by $N(N-1)/(\pi^2 L^4)$, as a function of the phase v of the Wilson line (when all eigenvalues coincide). There are two minima at $v = \pi/2$ and $3\pi/2$. These two minima are exchanged by charge conjugation \mathcal{C} , indicating that \mathcal{C} breaks spontaneously in QCD(AS) on $\mathbb{R}^3 \times S^1$ when the circumference L is small compared to Λ^{-1} . For $\mathcal{N} = 1$ SYM, the one loop potential identically vanishes.

If $L \ll \Lambda^{-1}$, so that $g^2(1/L) \ll 1$, then higher order corrections to the effective potential are small perturbations which cannot change the conclusion that the effective potential has two distinct gauge-inequivalent minima, related by charge conjugation. This shows that QCD(AS) on $\mathbb{R}^3 \times S^1$, for sufficiently small radius, spontaneously breaks charge conjugation symmetry. The imaginary part of the trace of the Wilson line is an order parameter for this symmetry. Its expectation value,

$$\left\langle \text{Im} \frac{\text{tr} \Omega}{N} \right\rangle = \pm i + O(g^2), \quad (3.7)$$

is nonzero and demonstrates spontaneous breaking of \mathcal{C} . The vacuum energy density equals the value of the effective potential at its minima and, to lowest order, is¹³

$$\mathcal{E} = -\frac{\pi^2}{24 L^4} N(N - \frac{7}{15}). \quad (3.8)$$

In contrast, the vacuum structure of the $\mathcal{N} = 1$ SYM theory, on $\mathbb{R}^3 \times S^1$, is essentially dictated by supersymmetry. At the perturbative level, to all orders, $V_{\text{eff}}(\Omega) = 0$, implying a compact moduli space of T^N/S_N on which the eigenvalues of Ω roam freely. [T^N is the maximal torus and S_N the Weyl group of $U(N)$.] Nonperturbatively, however, the moduli

¹³ $\Omega = \pm i$ is the correct vacuum configuration for $U(N)$ QCD(AS) theory for any value of N . Careful readers will notice that this configuration is only an element of $SU(N)$ when N is a multiple of 4. For $SU(N)$ theories with $N \bmod 4 \neq 0$, vacuum configurations are those elements of the center of $SU(N)$ which lie closest to $\pm i$. For large N , the resulting vacuum energy difference between $SU(N)$ and $U(N)$ is of order one, and hence subleading relative to the $O(N^2)$ result (3.8). Charge conjugation symmetry in $SU(N)$ QCD(AS) (on $\mathbb{R}^3 \times S^1$ with small radius) is spontaneously broken for all values of $N > 2$.

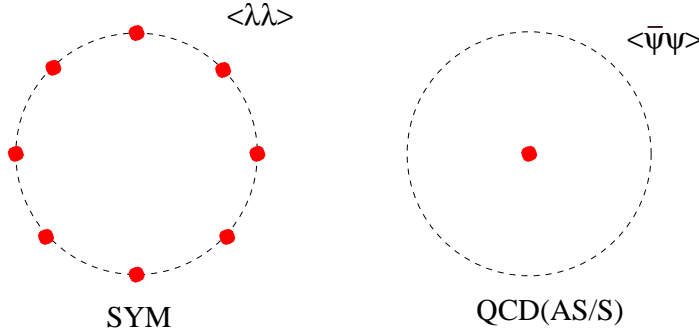


Figure 3: Chiral condensates of $\mathcal{N} = 1$ SYM (left) and QCD(AS/S) (right), for $L \ll \Lambda^{-1}$. The $\mathcal{N} = 1$ SYM theory has N chirally asymmetric vacuum states in which the fermion condensate $\langle \lambda\lambda \rangle$ has a non-zero magnitude and a phase equal to an integer multiple of $2\pi/N$. In contrast, QCD(AS/S) has a chirally symmetric phase with vanishing chiral condensate.

space is lifted by instanton and monopole contributions [37]. A nonperturbative superpotential is generated, and it provides a repulsive interaction between eigenvalues. Consequently, as shown in a nice treatment by Davies *et al.* [38, 39], in the vacuum state, the eigenvalues of the Wilson line are distributed equidistantly around the unit circle. Both the vacuum energy, and $\langle \text{tr} \Omega \rangle$ vanish identically. Charge conjugation symmetry [as well as the $U(1)$ center symmetry] is unbroken in $\mathcal{N} = 1$ SYM for all values of the S^1 circumference.

Chiral condensates

It is also instructive to compare the chiral properties of QCD(AS/S) and $\mathcal{N} = 1$ SYM. The case of $\mathcal{N} = 1$ SYM is well known [32]. The theory has a \mathbb{Z}_{2N} chiral symmetry which is the non-anomalous discrete remnant of the anomalous $U(1)_R$ symmetry. This \mathbb{Z}_{2N} chiral symmetry is spontaneously broken down to the \mathbb{Z}_2 of $(-1)^F$ by the formation of a fermion bilinear condensate. As a result, the theory has N isolated, supersymmetric vacua. This symmetry realization is independent of the size of the S^1 circle, and remains valid in the decompactified $L \rightarrow \infty$ limit. Each of the N vacua are invariant under a suitably redefined charge conjugation symmetry (equal to \mathcal{C} times one of the elements of the \mathbb{Z}_{2N} chiral symmetry).

QCD(AS) has an analogous \mathbb{Z}_{2N-4} non-anomalous discrete chiral symmetry [and QCD(S) has a \mathbb{Z}_{2N+4} chiral symmetry]. (See, for example, Ref. [34].) On $\mathbb{R}^3 \times S^1$ with small circumference, $L \ll \Lambda^{-1}$, this chiral symmetry is unbroken, unlike the case for $\mathcal{N} = 1$ SYM. To see this, note that the value $\Omega = \pm i$ for the Wilson line, corresponding to one of the minima of the effective potential discussed above, is equivalent to a constant background gauge field $A_4 = \pm\pi/(2L)$ (with x_4 labeling the compactified direction). This has the effect of shifting the allowed frequencies in the mode decomposition of antisymmetric representation fermions from the usual integer multiples of $2\pi/L$, appropriate for periodic boundary conditions and no gauge field, to integers plus a half, $\omega_n = 2\pi(n + \frac{1}{2})/L$, $n = 0, \pm 1, \pm 2, \dots$. (It is a shift by $1/2$, not $1/4$, because the fermions are in a two-index representation.) Hence, all allowed frequencies are non-zero and bounded below, in magnitude, by π/L . When analyzing dynamics

at distances large compared to L , the fermion field may be viewed as a Kaluza-Klein tower of three-dimensional fields, with effective masses equal to $|\omega_n|$. All modes act like very heavy fields and may be integrated out perturbatively, with no formation of any non-perturbative fermion condensate and consequently no breaking of chiral symmetry.¹⁴

Therefore, when the compactification size L is small compared to Λ^{-1} , QCD(AS/S) spontaneously breaks charge conjugation symmetry but does not break discrete chiral symmetry, while $\mathcal{N}=1$ SYM does exactly the opposite, breaking chiral symmetry but not charge conjugation. This difference in chiral symmetry realizations is illustrated in Fig. 3.

Symmetry realizations and order parameters

In addition to charge conjugation (whose realization determines whether there is a useful large N equivalence between QCD(AS/S) and SYM) and discrete chiral symmetry, it is also instructive to consider the realization of other global symmetries. Compactifying one direction preserves the parity invariance which is a symmetry of both $\mathcal{N}=1$ SYM and QCD(AS/S). The compactification also creates a new global symmetry known as center symmetry.¹⁵ For $U(N)$ SYM, the center symmetry is a continuous $U(1)$ invariance. For $U(N)$ QCD(AS/S), the presence of fermions with N -ality two reduces the center symmetry to \mathbb{Z}_2 , corresponding to invariance under gauge transformations which are antiperiodic in the compactified direction. For the following discussion, let \mathcal{Z} denote an antiperiodic center symmetry transformation.

Both charge conjugation \mathcal{C} and parity \mathcal{P} (or $x \rightarrow -x$ reflection) map the Wilson line to its conjugate, $\Omega \rightarrow \Omega^\dagger$, while the center symmetry transformation \mathcal{Z} of QCD(AS/S) negates the Wilson line, $\Omega \rightarrow -\Omega$. Therefore, the imaginary part of the trace of the Wilson line, $\text{Im}(\text{tr } \Omega)$, is odd under each of the three symmetry transformations \mathcal{C} , \mathcal{P} , and \mathcal{Z} , but is even under the product of any two of these transformations. Consequently, when the expectation value of $\text{Im}(\text{tr } \Omega)$ is non-zero, this corresponds to a symmetry breaking pattern in which the $(\mathbb{Z}_2)^3$ symmetry group generated by \mathcal{C} , \mathcal{P} , and \mathcal{Z} is spontaneously broken down to the $(\mathbb{Z}_2)^2$

¹⁴Another way to understand this is to note that fermion propagators fall exponentially like $e^{-\pi|\mathbf{x}|/L}$ at large distances, $|\mathbf{x}| \gg L$. Therefore, fermionic correlations are exponentially small at distances where the gauge field fluctuations become strongly coupled. Chiral symmetry breaking, if it were to occur, would be signaled by the failure of cluster decomposition in the chirally-invariant functional integral measure, $\lim_{|\mathbf{x}| \rightarrow \infty} \langle \mathcal{O}(\mathbf{x})^\dagger \mathcal{O}(0) \rangle \neq 0$ for $\mathcal{O}(\mathbf{x})$ a chiral symmetry order parameter such as $\bar{\psi}(\mathbf{x})\psi(\mathbf{x})$. Such a correlator must involve two (or more) fermion propagators connecting the two operators. If fermion propagators are exponentially small at large distance, then this violation of cluster decomposition cannot possibly occur. For a rigorous proof along these lines, in the context of high temperature lattice gauge theory, see Refs. [40, 41].

¹⁵Let \hat{x} denote the compactified direction (with period L). Center symmetry is the invariance of a gauge theory under gauge transformations $g(x)$ which are only periodic up to an element of the center of the gauge group, $g(x + L\hat{x}) = \omega g(x)$, with ω some element of the gauge group (other than the identity) which commutes with all group elements. Although most easily described as invariance under aperiodic gauge transformations, center symmetry should be regarded as a *global* symmetry of the theory. Physical states need not be invariant under non-trivial center symmetry transformations; Gauss' law only requires that physical states be invariant under periodic gauge transformations (which are continuously connected to the identity). The global center symmetry is really the quotient of the full gauge group (including aperiodic gauge transformations) divided by the subgroup of periodic transformations. Since all physical states are invariant under periodic gauge transformations, one may identify non-trivial center symmetry transformations which only differ by a periodic gauge transformation.

subgroup consisting of \mathcal{CP} , \mathcal{CZ} , and \mathcal{PZ} (plus the identity), resulting in two degenerate vacua, as seen in the above analysis of the effective potential.

The only operators which can function as order parameters for this particular symmetry breaking pattern must, like $\text{Im}(\text{tr} \Omega)$, be odd under each of \mathcal{C} , \mathcal{P} , and \mathcal{Z} . Being odd under only one or two of these transformations is not sufficient.¹⁶ Consequently, generic \mathcal{C} -odd operators, such as the imaginary part of topologically trivial Wilson loops, cannot reveal this symmetry breaking pattern. Only non-local \mathcal{C} , \mathcal{P} , and \mathcal{Z} odd operators, such as $\text{Im}(\text{tr} \Omega^k)$, for k odd, will have expectation values which reveal the symmetry breaking in QCD(AS/S) at small radius.

There is one more symmetry worth discussing, namely time-reversal \mathcal{T} . This transformation leaves invariant the trace of the spacelike Wilson,¹⁷ $\mathcal{T}(\text{tr} \Omega) \mathcal{T}^{-1} = \text{tr} \Omega$. Hence one might think that an imaginary expectation value for $\text{tr} \Omega$ would have no implications for the realization of time reversal symmetry. This is incorrect, however, because time-reversal is an anti-unitary transformation. If one decomposes $\text{tr} \Omega$ into real and imaginary parts, then $\text{Re}(\text{tr} \Omega) = \frac{1}{2} [\text{tr} \Omega + \text{tr} \Omega^\dagger]$ is time-reversal even, while the Hermitian operator $\text{Im}(\text{tr} \Omega) = \frac{1}{2i} [\text{tr} \Omega - \text{tr} \Omega^\dagger]$ is necessarily time-reversal odd. Unbroken time reversal symmetry implies that any Hermitian, time-reversal odd operator must have vanishing expectation value. Consequently, in addition to signaling spontaneous symmetry breaking of charge conjugation, parity, and center symmetry, the non-zero expectation value of $\text{Im}(\text{tr} \Omega)$ in QCD(AS/S), at small radius, also implies spontaneous breaking of time reversal (and hence also CPT) symmetry. But, as noted above, the only observables which are sensitive to this symmetry breaking are operators involving topologically non-trivial Wilson loops with odd winding numbers around the compactified direction.

High temperature

Changing the fermion boundary conditions on the S^1 from periodic to antiperiodic allows the theory on $\mathbb{R}^3 \times S^1$ to be reinterpreted as a thermal field theory on \mathbb{R}^3 space with inverse temperature $\beta = L$. The Wilson line $\Omega(\mathbf{x})$ wrapping the S^1 is now a thermal Polyakov loop, and serves as an order parameter for the center symmetry of the theory.¹⁸ Once again, if the temperature $T = L^{-1}$ is large compared to the confinement scale Λ , then a perturbative calculation of the effective potential for the Wilson line is reliable. For QCD(AS/S), changing the boundary conditions for the fermions is exactly equivalent to replacing Ω by $i\Omega$ in the

¹⁶An order parameter must transform non-trivially under at least some of the spontaneously broken symmetries while being invariant under the unbroken symmetry group. If an operator \mathcal{O} is invariant under any one of the three transformations \mathcal{C} , \mathcal{P} , and \mathcal{Z} , and is also invariant under the unbroken \mathcal{CP} , \mathcal{CZ} and \mathcal{PZ} transformations, then it is necessarily invariant under each of \mathcal{C} , \mathcal{P} , and \mathcal{Z} .

¹⁷For reference, if the gauge field is regarded as an anti-Hermitian matrix and the metric signature is $(-+++)$, then the actions of \mathcal{C} , \mathcal{P} and \mathcal{T} transformations on the gauge field are given by $A_\mu(t, \mathbf{x}) \xrightarrow{\mathcal{C}} +A_\mu(t, \mathbf{x})^*$, $A_\mu(t, \mathbf{x}) \xrightarrow{\mathcal{P}} -A^\mu(t, -\mathbf{x})$, and $A_\mu(t, \mathbf{x}) \xrightarrow{\mathcal{T}} +A^\mu(-t, \mathbf{x})$.

¹⁸When the compactified direction is regarded as Euclidean time (with antiperiodic boundary conditions for fermions), then the center symmetry realization determines whether the theory is in a confining or deconfined phase. Unbroken center symmetry means a confining phase, while spontaneous breaking of center symmetry indicates a deconfined phase [42, 43].

result (3.4) for the effective potential, or equivalently shifting the eigenvalues $v_i \rightarrow v_i + \pi/2$. (Switching the sign in the boundary conditions corresponds to a shift of $\pi/2$, not π , because the fermions are in a two index representation.) Hence, the one-loop potential for QCD(AS) becomes

$$V_{\text{eff}}^{\text{QCD(AS)}}(\Omega) = \frac{T^4}{24\pi^2} \left\{ \sum_{i,j=1}^N [v_i - v_j]^2 ([v_i - v_j] - 2\pi)^2 - \frac{8}{15} \pi^4 N \right. \\ \left. - 2 \sum_{i < j=1}^N [v_i + v_j + \pi]^2 ([v_i + v_j + \pi] - 2\pi)^2 \right\}. \quad (3.9)$$

This is now minimized when all $v_i = 0$, or all $v_i = \pi$, or in other words, when $\Omega = \pm 1$. This demonstrates spontaneous breaking of the \mathbb{Z}_2 center symmetry, and indicates that the high temperature phase of the theory is a deconfined phase. The value of this potential at its minimum gives the leading order free energy density (or minus the pressure),

$$\mathcal{F}^{\text{QCD(AS)}} = -\frac{\pi^2}{24} T^4 N(N - \frac{7}{15}), \quad (3.10)$$

which equals the expected Stefan-Boltzmann result of $-\frac{\pi^2}{45} T^4 [N^2 + \frac{7}{8} N(N-1)]$.

The non-vanishing Polyakov loop, $\Omega = \pm 1$, and doubly degenerate minima signal spontaneous breaking of the \mathbb{Z}_2 center symmetry of QCD(AS). But because $\langle \text{tr } \Omega \rangle$ is now real, there is no longer any spontaneous breaking of charge conjugation symmetry (or parity or time-reversal) in this high temperature deconfined phase.¹⁹ Chiral symmetry is also unbroken in this phase.

Since charge conjugation symmetry is unbroken in this hot plasma phase, the large- N equivalence to $\mathcal{N} = 1$ SYM, within the neutral sector, is guaranteed to be applicable to thermal expectation values and connected correlators.²⁰ However, it may be instructive to see this explicitly in a simple example, such as the large N free energy.

Thermal (antiperiodic) boundary conditions for the fermions break supersymmetry and cause SYM to develop a non-vanishing effective potential for the Polyakov loop. Evaluating the required functional determinant for adjoint representation fermions with antiperiodic boundary conditions, as described in the Appendix, leads to

$$V_{\text{eff}}^{\text{SYM}}(\Omega) = \frac{T^4}{24\pi^2} \left\{ \sum_{i,j=1}^N [v_i - v_j]^2 ([v_i - v_j] - 2\pi)^2 - \sum_{i,j=1}^N [v_i - v_j + \pi]^2 ([v_i - v_j + \pi] - 2\pi)^2 \right\}. \quad (3.11)$$

¹⁹Although we have only performed a perturbative analysis, these conclusions are clearly valid non-perturbatively. As for ordinary QCD, the equilibrium long-distance non-perturbative dynamics of high temperature QCD(AS/S) is described by a three-dimensional effective theory which is just $3d$ Yang-Mills. This theory has a unique ground state and no symmetry breaking.

²⁰In the case of the nonsupersymmetric \mathbb{Z}_2 orbifold projection of $\mathcal{N} = 1$ SYM [16], precisely the same considerations reveal that the \mathbb{Z}_2 symmetry permuting gauge group factors is unbroken in the high temperature, deconfined phase of the theory,. Therefore, large- N equivalence to $\mathcal{N} = 1$ SYM is applicable within this phase of the theory.

This is minimized when all eigenvalues coincide, or when $\Omega = e^{iv} \mathbb{1}$ for any phase v . This demonstrates spontaneous breaking of the $U(1)$ center symmetry in hot SYM. The resulting leading order free energy density is

$$\mathcal{F}^{\text{SYM}} = -\frac{\pi^2}{24} T^4 N^2, \quad (3.12)$$

which is the expected Stefan-Boltzmann result for this theory. As required by large N equivalence, the leading $O(N^2)$ piece of the QCD(AS) free energy (3.10) coincides with the SYM result (3.12). At this lowest order, the comparison is essentially trivial, and just reflects that both theories have, to order N^2 , the same number of bosonic and fermionic degrees of freedom. But the large- N equivalence guarantees that the $O(N^2)$ parts of the free energy, in the high temperature phase, will coincide exactly.

Spontaneous breaking of the center symmetry generates an uncountable number of extremal equilibrium states (labeled by the phase v) in $U(N)$ SYM, since the center symmetry is a continuous $U(1)$, but only two extremal states in QCD(AS/S), where the center symmetry is a discrete \mathbb{Z}_2 . Only for the two states of hot SYM with $\langle \text{tr } \Omega / N \rangle = \pm 1$ do the Polyakov loop expectation values coincide with QCD(AS/S). This is completely consistent with the large N equivalence, as these are the only extremal equilibrium states of SYM which lie within the neutral (charge conjugation invariant) sector to which the large N equivalence applies.

Compactification on $T^3 \times \mathbb{R}$

Instead of compactifying a single direction one may, of course, choose to compactify two or more directions. A particular case, which has been previously considered by Barbon and Hoyos [26], is compactification on $T^3 \times \mathbb{R}$, with a symmetric three torus and periodic boundary conditions for the fermions in all directions. For this finite volume spatial compactification, all dynamics is weakly coupled if the physical size of the three torus is much smaller than confinement scale Λ^{-1} . For QCD(AS) (as well as the \mathbb{Z}_2 orbifold projection of SYM), Ref. [26] analyzed the resulting one-loop effective potential for Wilson lines. This corresponds to a Born-Oppenheimer approximation in the quantum mechanics of these finite volume theories.

The discussion in Ref. [26] was focused on trying to identify properties of these non-supersymmetric theories which might somehow be related, at large N , to the supersymmetric index of SYM. It did not directly address the much more basic issue of the symmetry realizations of these theories in the $N \rightarrow \infty$ limit.

Examination of the effective potential found in Ref. [26] for QCD(AS) shows that it has eight-fold degenerate global minima corresponding to Wilson lines along each of the three directions of the torus equaling $\pm i$. In the $N \rightarrow \infty$ limit, this is properly interpreted as indicating spontaneous breaking of both the $(\mathbb{Z}_2)^3$ center symmetry on the torus, and charge conjugation symmetry.²¹ In other words, the status of large N orientifold (or orbifold) equivalence is exactly the same on $T^3 \times \mathbb{R}$ with small volume and $\mathbb{R}^3 \times S^1$ with small radius.

²¹For finite (spatial) volume and finite values of N , the theory must have a unique ground state and no spontaneous symmetry breaking (just like a generic quantum theory with a finite number of degrees of freedom). But the $N \rightarrow \infty$ limit is a thermodynamic limit, just like the infinite volume limit of a typical statistical system, and spontaneous symmetry breaking in the $N = \infty$ limit is perfectly possible. The tunneling amplitudes

In either case, symmetry breaking in the non-supersymmetric daughter theory (with periodic boundary conditions) prevents any useful large N equivalence to SYM.

4. Phase diagrams

Examination of the one loop effective potential has taught us that charge conjugation symmetry is spontaneously broken in QCD(AS/S) (at zero temperature) when one spatial dimension is compactified with size $L \ll \Lambda^{-1}$. However, this perturbative analysis can not tell us anything about symmetry realizations when $L \gg \Lambda^{-1}$. No proof demonstrating the absence (or presence) of spontaneous breaking of charge conjugation in the strongly coupled large radius regime is known. For sufficiently large radius (or in the decompactified theory on \mathbb{R}^4), QCD(AS) [or QCD(S)] is expected to develop a chiral condensate, $\langle \bar{\psi}\psi \rangle \neq 0$, and thereby spontaneously break its discrete chiral symmetry down to the \mathbb{Z}_2 of $(-1)^F$. This is not unequivocally established, but is consistent with the effects of instanton induced interactions, and is supported by the results of the simulations reported in Ref. [30]. Assuming this is the case, then QCD(AS/S) on $\mathbb{R}^3 \times S^1$ must have a chiral symmetry breaking phase transition at some critical size L_χ (which would necessarily be of order Λ^{-1}).

If the chirally-asymmetric large radius phase has unbroken charge conjugation symmetry, then within this phase the large N equivalence connecting QCD(AS/S) and $\mathcal{N}=1$ SYM is useful and, as discussed in Refs. [21–24], generates interesting quantitative predictions such as equality of chiral condensates. The simplest scenario is that there is a single phase transition at a critical size L_c where both chiral symmetry and charge conjugation realizations change. But it is also possible that there are two distinct transitions, at radii L_χ and L_c , with chiral symmetry breaking for $L > L_\chi$ and broken charge conjugation symmetry for $L < L_c$. If so, then the large- N equivalence to SYM implies that L_χ must be less than L_c (for large N), since otherwise the vanishing of the chiral condensate in the interval $L_c < L < L_\chi$ would contradict the large- N equivalence to SYM. If there are two distinct transitions, then the intermediate phase for $L_\chi < L < L_c$ would have both chiral symmetry and charge conjugation spontaneously broken, and no useful large- N equivalence to SYM.

It is also logically possible that that charge conjugation symmetry remains broken for all radii, in which case QCD(AS) [or QCD(S)] would have a single phase transition and a large radius phase in which both discrete chiral symmetry and charge conjugation symmetry are spontaneously broken. In this case, the large N equivalence to $\mathcal{N}=1$ SYM, which is valid

between different minima of the effective potential decrease exponentially with increasing N , so the lifetime of a state localized near a single minimum diverges as $N \rightarrow \infty$.

More formally, one may test for spontaneous symmetry breaking in either of two ways. One may add a symmetry breaking perturbation of strength ϵ to the theory and test whether the $\epsilon \rightarrow 0$ limit of the expectation value of an order parameter, after sending $N \rightarrow \infty$, depends on the direction of approach of ϵ to zero. Alternatively, one may leave the theory unchanged, so that the ground state is completely symmetric, and instead test whether expectation values of products of order parameters, in the $N \rightarrow \infty$ limit, satisfy large N factorization. This corresponds to testing cluster decomposition in the usual large volume limit, and is an equally valid indicator of symmetry breaking. Failure of large N factorization (or cluster decomposition) implies that, in the thermodynamic limit, the symmetric ground state is indistinguishable from a statistical mixture of extremal pure states which do satisfy factorization (or cluster decomposition).

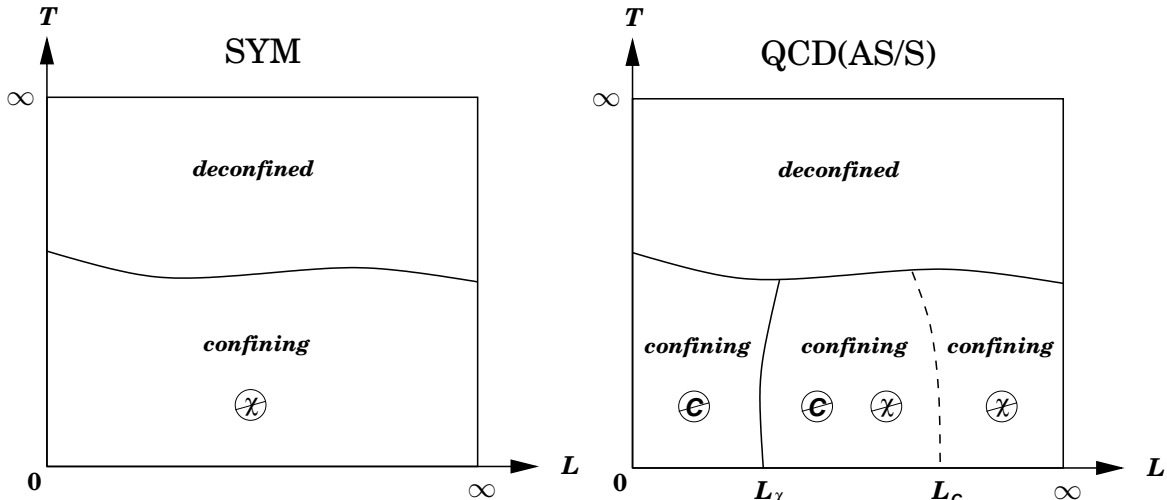


Figure 4: Schematic phase diagram of $U(N)$ $\mathcal{N}=1$ SYM (left) and QCD(AS/S) (right) as a function of temperature T and compactification size L of one spatial direction. $\mathcal{N}=1$ super-Yang-Mills has a confining low temperature phase in which the discrete chiral symmetry is spontaneously broken, and a deconfined high temperature phase with unbroken chiral symmetry. Charge conjugation is unbroken in both phases. QCD(AS/S) has multiple confining low temperature phases. For sufficiently small radius, $L < L_c$, it spontaneously breaks charge conjugation symmetry. For sufficiently large radius, $L > L_\chi$, it is believed to break spontaneously its discrete chiral symmetry. As discussed in the text, the intermediate phase, with both charge conjugation and chiral symmetry breaking, may not exist, or it may extend all the way to $L = \infty$. The deconfined high temperature phase has unbroken chiral and charge conjugation symmetry. Large N equivalence to SYM applies only within the phases of QCD(AS/S) with unbroken charge conjugation symmetry. These sketches depict the simplest scenario in which only a single phase transition separates high and low temperature phases; more complicated scenarios with distinct deconfinement, chiral restoration, and [for QCD(AS/S)] charge conjugation restoration transitions are also possible.

only in the neutral sectors of these theories, would not be useful for predicting low energy properties since the ground state of QCD(AS) would always lie outside this sector.

Examining these theories as a function of temperature and the compactified spatial radius reveals a richer phase structure. (By this we mean considering these theories on $\mathbb{R}^2 \times (S^1)^2$, with one S^1 having circumference L and periodic boundary conditions for the fermions, and the other S^1 having circumference $\beta = 1/T$ and antiperiodic boundary conditions.) As discussed in the previous section, at sufficiently high temperatures both SYM and QCD(AS/S) must have a deconfined plasma phase in which chiral and charge conjugation symmetries are unbroken. (If $\beta \ll L$, then the compactification of a spatial direction is not significant in the analysis of high temperature thermodynamics. Hence, the discussion of high temperature in the last section applies equally well to this regime of two compactified directions.)

In $\mathcal{N}=1$ SYM, for any value of the spatial size L , as one raises the temperature there must be at least one phase transition separating the confining, chirally asymmetric low temperature phase from the chirally symmetric high temperature plasma phase. [The $U(1)$ center symmetry is spontaneously broken in the plasma phase, and unbroken in the confining phase.] Whether there is a single phase transition, at which both confinement and chiral symmetry realizations change, or two phase transitions separating a distinct intermediate phase is not

clear. We are unaware of any evidence indicating which alternative applies.

In QCD(AS/S), all the possible zero temperature phases discussed above are confining phases. Starting in any of these phases, as the temperature is increased there must again be one or more phase transitions separating the low temperature phase from a high temperature plasma phase. (The \mathbb{Z}_2 center symmetry of the theory is spontaneously broken in the plasma phase, and unbroken in the confining phase.) For some values of L , there could be transitions at three distinct temperatures corresponding to separate chiral symmetry restoration, charge conjugation restoration, and deconfinement transitions.

Fig. 4 depicts the simplest possibility, in which a single transition separates the low temperature phases from the high temperature plasma phase. But as just noted, much more complicated possibilities might occur. Large- N equivalence between QCD(AS/S) and $\mathcal{N} = 1$ SYM (for equilibrium correlators of neutral operators) is valid in any phase of QCD(AS/S) which does not spontaneously break charge conjugation symmetry — and only in such phases. Hence, the equivalence is valid within the deconfined plasma phase. And it also applies within the large radius low temperature phase with (only) broken chiral symmetry — if this phase exists.²²

It should be emphasized that the preceding discussion, and Fig. 4, are specifically addressing the case of $U(N)$ gauge theories and, for QCD(AS), tacitly assume that $N > 3$. If $N = 3$, then QCD(AS) has no chiral symmetry other than the \mathbb{Z}_2 of $(-1)^F$, and hence no chiral symmetry breaking. And (as noted in footnote 13) spontaneous breaking of charge conjugation (and \mathcal{P} and \mathcal{T}) symmetry at small radius only occurs for $N > 2$.²³

²²The discussion of this section is equally applicable to the \mathbb{Z}_2 orbifold projection of $\mathcal{N} = 1$ SYM yielding a non-supersymmetric $U(N) \times U(N)$ gauge theory with a bifundamental fermion. Charge conjugation symmetry in that case is replaced by the \mathbb{Z}_2 symmetry interchanging the two gauge group factors. With this substitution, the sketch of the phase diagram of QCD(AS/S) in Fig. 4 is equally appropriate for the \mathbb{Z}_2 orbifold case.

²³Replacing the $U(N)$ gauge group by $SU(N)$ eliminates the $U(1)$ photon from the theory. In the large N limit, this change is irrelevant. But for finite N there are some important differences. For SYM theory, the $U(1)$ gauge field is decoupled from all other degrees of freedom, for any N , and removing it has no effect of the dynamics or the phase diagram. [Switching from $U(N)$ to $SU(N)$ in SYM does change the center symmetry from $U(1)$ to \mathbb{Z}_N . In both cases the center symmetry is spontaneously broken in the high temperature phase of the theory. This is true even when one spatial dimension is compactified, so that the long distance physics is effectively two dimensional. Hot $U(N)$ SYM with its free $U(1)$ gauge field illustrates the fact that one can formally have spontaneous symmetry breaking of a continuous symmetry in two dimensions if the Goldstone bosons are decoupled.] But for $SU(N)$ QCD(AS/S), only if N is even does the theory have a \mathbb{Z}_2 center symmetry, whose realization provides a sharp distinction between confined and deconfined phases. There is no center symmetry if N is odd, reflecting the fact that, in this case, (anti)symmetric representation fermions can screen test charges in any representation. Assume, for the sake of discussion, that spontaneous breaking of charge conjugation symmetry does not survive to arbitrarily large radius, so that L_C in Fig. 4 is finite. If one raises the temperature starting in the confining large volume phase of $SU(N)$ QCD(AS/S) then, for N even, there must be a sharp deconfinement transition even in the presence of a non-zero fermion mass (which eliminates the chiral symmetry and, for sufficiently large mass, can eliminate a chiral phase transition). (See also Ref. [44].) But for N odd, there need be no phase transition associated with deconfinement. The massless theory [with $N > 3$ for QCD(AS)] must still have a sharp chiral transition, but at non-zero fermion mass this can turn into a smooth crossover.

$SU(3)$ QCD(AS) is a special case with no center symmetry and no chiral symmetry [other than $(-1)^F$]. In this theory, for any fermion mass, there can be a smooth crossover with no sharp transitions separating the

5. Multiple fermion flavors

Instead of considering theories with just one fermion field, one may generalize the entire previous discussion to the case of multiple fermion flavors. Starting with a non-supersymmetric $SO(2N)$ Yang-Mills with n_f adjoint representation Majorana fermions, orientifold projections by J , or $J(-1)^F$, yield $U(N)$ gauge theories with either n_f adjoint representation Majorana fermions, or n_f antisymmetric tensor representation Dirac fermions, respectively. (To preserve asymptotic freedom in these theories, n_f must be at most five.) Once again, there is a large- N equivalence between these two daughter theories within their respective neutral sectors (corresponding to bosonic, charge conjugation even operators).

It is completely straightforward to generalize the analysis of section 3 to multiple (massless) flavors. When compactified on $\mathbb{R}^3 \times S^1$ with sufficiently small radius, and periodic boundary conditions, one finds that the one loop effective potential for the Wilson line in $U(N)$ Yang-Mills with n_f adjoint fermions generates a repulsive interaction between eigenvalues. Consequently, the Wilson line eigenvalues distribute uniformly around the unit circle and $\langle \text{tr } \Omega \rangle = 0$. Charge conjugation symmetry is unbroken. This theory has a non-anomalous $SU(n_f) \times \mathbb{Z}_{2Nn_f}$ chiral symmetry which is expected to break down to $SO(n_f)$ by the formation of a fermion bilinear condensate, giving rise to a vacuum manifold with N disjoint components, each of which is the coset space $SU(n_f)/SO(n_f)$. We expect this chiral symmetry realization to hold for all values of the S^1 circumference. The same analysis with antiperiodic boundary conditions shows that at sufficiently high temperatures the theory has spontaneously broken center symmetry but unbroken chiral and charge conjugation symmetry, just like hot SYM.

The situation for QCD(AS) with $n_f > 1$ flavors is essentially the same as for $n_f = 1$. On $\mathbb{R}^3 \times S^1$ with sufficiently small radius, charge conjugation symmetry is spontaneously broken. Hence, the ground state does not lie in the neutral sector and large N equivalence to Yang-Mills with adjoint fermions is not applicable to vacuum expectation values. Also, within this phase of the theory, the $SU(n_f)_L \times SU(n_f)_R \times \mathbb{Z}_{(2N-4)n_f}$ chiral symmetry remains unbroken. For sufficiently large radius, we expect the chiral symmetry to be spontaneously broken down to $SU(n_f)_V$ leading to a vacuum manifold with $N - 2$ components each of which is the coset space $[SU(n_f)_L \times SU(n_f)_R]/SU(n_f)_V$. At sufficiently high temperature, it is again easy to establish that charge conjugation and chiral symmetries are unbroken, and large N equivalence to Yang-Mills with adjoint fermions, within the neutral sector, is applicable to this phase.

The realization of charge conjugation symmetry, at large radius (and zero temperature), is not currently known. Let us assume, for the sake of discussion, that charge conjugation symmetry is unbroken for sufficiently large radius. Then large N equivalence between the multiflavor adjoint and antisymmetric representation theories will be applicable to their confining large radius phases. One point to note is that the number of Goldstone bosons does *not* match between these two theories. For Yang-Mills with n_f adjoint fermions, there are

confining large volume low temperature phase and the high temperature plasma. This is consistent with the numerical results of Refs. [30, 45]. It is not clear if this sensitivity of symmetry realizations to the value of N , for $SU(N)$ gauge groups, will be reflected in other properties of QCD(AS) (such as particle spectra).

$\frac{1}{2}n_f(n_f+1)-1$ Goldstone bosons while QCD(AS) with n_f flavors has n_f^2-1 . This is not in conflict with large N equivalence between these theories, as only $\frac{1}{2}n_f(n_f+1)-1$ of the Goldstone bosons of multiflavor QCD(AS) are charge conjugation even — and this correctly coincides with the number of Goldstone bosons in the multiflavor adjoint representation theory (all of which are \mathcal{C} even).

6. Remarks

Orbifolds versus orientifolds, and kinematics versus dynamics

One of the motivations in Ref. [21] for considering the orientifold equivalence between SYM and QCD(AS/S) was the belief that twisted sectors were absent, coupled with a belief that the mere existence of a twisted sector would spoil large N equivalence [21, 23]. (These points are also emphasized in the recent work [46].) However, for either orbifold or orientifold equivalences, the presence of twisted sectors is an inevitable consequence of the projections by discrete symmetries which define the mappings between theories. As we have emphasized, what is significant for large N equivalence is not the existence of twisted sectors, but rather the realization of the symmetries which define the neutral and twisted sectors. In this regard, orbifold and orientifold equivalences are on exactly the same footing.

As illustrated by the specific example of QCD(AS/S), it is also inevitable that the symmetry realization of a theory depends on its specific dynamics and on the phase of the theory under consideration. In much of the discussion in the literature concerning “tests” of large N orbifold or orientifold equivalence, the focus has been on kinematic aspects of the mapping between theories, without addressing the more serious issue of the influence of dynamics on the symmetry realization.

It is important to understand that large N equivalence between theories does not mean equality. Rather, it means that there is a well defined mapping connecting a specific class of (neutral) observables in the two theories. In some cases, such as orbifold projections which change the dimension of the gauge group, the correct mapping between theories requires appropriate rescalings of operators and correlation functions [15, 16]. Misunderstanding of the correct mapping between theories, or of the limitation of the equivalence to appropriate neutral sectors, has led to several unjustified claims of inequivalence.

String theory realizations

It has been argued that large N equivalence between $\mathcal{N}=1$ SYM and QCD(AS/S) may naturally be understood in the context of non-tachyonic string theory, and that this ensures the validity of the orientifold equivalence. (See pages 71–80 of Ref. [23] and references therein, as well as section 7 of Ref. [46].) This is an interesting argument, especially in light of our demonstration that orientifold equivalence *can* fail due to spontaneous breaking of charge conjugation symmetry. The essence of this argument boils down to three steps: *(i)* the confining, asymptotically free gauge theories of interest [$\mathcal{N}=1$ SYM and QCD(AS/S)] have string theory duals, *(ii)* the dual string theory realizations of these theories are, by construction, tachyon free, and *(iii)* any symmetry-breaking instability in field theory would necessarily appear as a tachyon in the string theory realization.

We believe these assertions overlook essential caveats which undermine this argument. There is no string theory realization of $\mathcal{N}=1$ SYM or QCD(AS/S) for which the string theory is under control in the decoupling limit needed to obtain precisely the non-gravitational theories of interest. (The recent paper [47] of Bena *et al.* is relevant in this regard.) Any string theory realization of an asymptotically free theory such as SYM and QCD(AS/S) will necessarily involve a highly curved background spacetime. The regime of the string theory where its dynamics is under control does not include the regime of interest. Therefore the presence, or absence, of tachyons in a weakly-coupled regime of the string theory, where it does not correspond to the desired field theory, tells one nothing about symmetry realizations in that field theory.

Summary

Large N “orientifold” equivalence between QCD(AS/S) and $\mathcal{N}=1$ SYM fails when the theories are compactified on small torii, due to spontaneous breaking of charge conjugation symmetry. The equivalence holds at sufficiently high temperature, when charge conjugation symmetry is unbroken. Whether charge conjugation symmetry remains spontaneously broken in large radius compactifications (or in the decompactified limit on \mathbb{R}^4) is not yet clear, and depends on detailed dynamics of the theory.

This situation is exactly parallel to the case of large N equivalence between $U(2N)$ $\mathcal{N}=1$ SYM and its \mathbb{Z}_2 orbifold projection yielding a $U(N) \times U(N)$ gauge theory with a bifundamental fermion [16]. This equivalence is valid at high temperature, but fails when the theories are compactified on sufficiently small torii, due to spontaneous breaking of the \mathbb{Z}_2 symmetry exchanging gauge groups in the daughter theory. Whether this symmetry is restored or remains broken in large volume is also not currently known.

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A. Functional determinants on $\mathbb{R}^3 \times S^1$

We need to evaluate $\ln \det_{\pm}(-D_{\mathcal{R}}^2)$, the logarithm of the functional determinant of the covariant Laplacian acting on $\mathbb{R}^3 \times S^1$ in the presence of an arbitrary constant $U(N)$ gauge field pointing in the compact direction, for various representations \mathcal{R} and with either periodic or antiperiodic boundary conditions on the S^1 (indicated by the subscript on the determinant). [A single Dirac fermion gives rise to $\det(\not{D}_{\mathcal{R}})$ which, since the gauge field strength vanishes, equals $\det^2(-D_{\mathcal{R}}^2)$. A Majorana fermion gives the Pfaffian (or square root of the determinant) of $\not{D}_{\mathcal{R}}$, which equals $\det(-D_{\mathcal{R}}^2)$.]

Working in a gauge in which the background gauge field (in the fundamental representation) is diagonal, with eigenvalues $\{v_i/L\}$ (with L the circumference of the S^1), makes it

easy to diagonalize the covariant Laplacian. To compute the log of the determinant, defined via dimensional continuation, the key ingredient is the identity

$$g(v) \equiv L^3 \sum_n \int \frac{d^3 k}{(2\pi)^3} \ln [k^2 + (2\pi n + v)^2 L^{-2}] = -\frac{\pi^2}{45} + \frac{1}{24\pi^2} [v]^2 (2\pi - [v])^2, \quad (\text{A.1})$$

where $[v] \equiv v \bmod 2\pi$. (For further details see, for example, appendix D of Ref. [36].)

Consequently, the functional determinant of the Laplacian in the fundamental representation is given by

$$\ln \det_+(-D_{\text{fund}}^2) = \frac{\mathcal{V}}{L^3} \sum_{i=1}^N g(v_i), \quad \ln \det_-(-D_{\text{fund}}^2) = \frac{\mathcal{V}}{L^3} \sum_{i=1}^N g(v_i + \pi), \quad (\text{A.2})$$

where \mathcal{V} is the spatial volume. For multi-index representations, one merely has to think how the gauge field, viewed as a diagonal $N \times N$ matrix, acts on the individual components of the representation. For the adjoint, symmetric tensor, and antisymmetric tensor representations, the appropriate generalizations are:

$$\ln \det_+(-D_{\text{adj}}^2) = \frac{\mathcal{V}}{L^3} \sum_{i,j=1}^N g(v_i - v_j), \quad \ln \det_-(-D_{\text{adj}}^2) = \frac{\mathcal{V}}{L^3} \sum_{i,j=1}^N g(v_i - v_j + \pi), \quad (\text{A.3})$$

$$\ln \det_+(-D_{\text{sym}}^2) = \frac{\mathcal{V}}{L^3} \sum_{i \leq j=1}^N g(v_i + v_j), \quad \ln \det_-(-D_{\text{sym}}^2) = \frac{\mathcal{V}}{L^3} \sum_{i \leq j=1}^N g(v_i + v_j + \pi), \quad (\text{A.4})$$

$$\ln \det_+(-D_{\text{antisym}}^2) = \frac{\mathcal{V}}{L^3} \sum_{i < j=1}^N g(v_i + v_j), \quad \ln \det_-(-D_{\text{antisym}}^2) = \frac{\mathcal{V}}{L^3} \sum_{i < j=1}^N g(v_i + v_j + \pi). \quad (\text{A.5})$$

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